

Week 5 - Friday

COMP 2230

Last time

- Induction examples
- Strong induction

Questions?

Assignment 2

Logical warmup

- A monk begins an ascent of Mt. Fuji on Monday morning, reaching the summit by nightfall.
- He spends the night at the summit and starts down the mountain on the same path the following morning, reaching the bottom by dusk on Tuesday.
- Show that at some precise time of day, the monk was at exactly the same spot on the path on Tuesday as he was on Monday.

Recursion

In order to understand recursion, you must first understand recursion.

Recursively defined sequences

- As you've seen with Fibonacci, it's possible to define a sequence recursively
- This is called a **recurrence relation**
- The **initial conditions** give the starting point
- Example:
 - Initial conditions
 - $c_0 = 1$
 - $c_1 = 2$
 - Recurrence relation
 - $c_k = c_{k-1} + kc_{k-2} + 1$, for all integers $k \geq 2$
 - Find c_2 , c_3 , and c_4

Writing recurrence relations in multiple ways

- Consider the following recurrence relation:
 - $s_k = 3s_{k-1} - 1$, for all integers $k \geq 1$
- Now consider this one:
 - $s_{k+1} = 3s_k - 1$, for all integers $k \geq 0$
- Both recurrence relations have the same meaning

Differences in initial conditions

- Even if the recurrence relations are equivalent, different initial conditions can cause a different sequence
- Example:
 - $a_k = 3a_{k-1}$, for all integers $k \geq 2$
 - $a_1 = 2$
 - $b_k = 3b_{k-1}$, for all integers $k \geq 2$
 - $b_1 = 1$
 - Find $a_1, a_2,$ and a_3
 - Find $b_1, b_2,$ and b_3

Practice

- Using just your wits, you should be able to figure out recursive definitions for many sequences
- Give a recurrence relation for positive even integers:
2, 4, 6, 8, ...
- Give a recurrence relation for the triangular numbers:
1, 3, 6, 10, 15, ...
- Give a recurrence relation for the perfect squares:
1, 4, 9, 16, 25, ...
- Give a recurrence relation for factorial: 1, 2, 6, 24, 120, 720 ...

The proof is on fire

- Perhaps you believe that you have the correct recurrence relation for perfect squares
- Can you prove it?
- **Hint:** Use mathematical induction
- Recursion and induction and two sides of the same coin
- The right cross and the left jab, if you will, of the computer scientist's arsenal

Tower of Hanoi

- Imagine that you have a tower of disks such that each is smaller than the one it rests on
- Rules:
 1. There are 3 pegs, and all the disks are on peg 1
 2. No two disks are the same size
 3. A larger disk may not be placed on a smaller disk
 4. Disks can only be moved one at a time
 5. Move all the disks to peg 3



Power of Hanoi

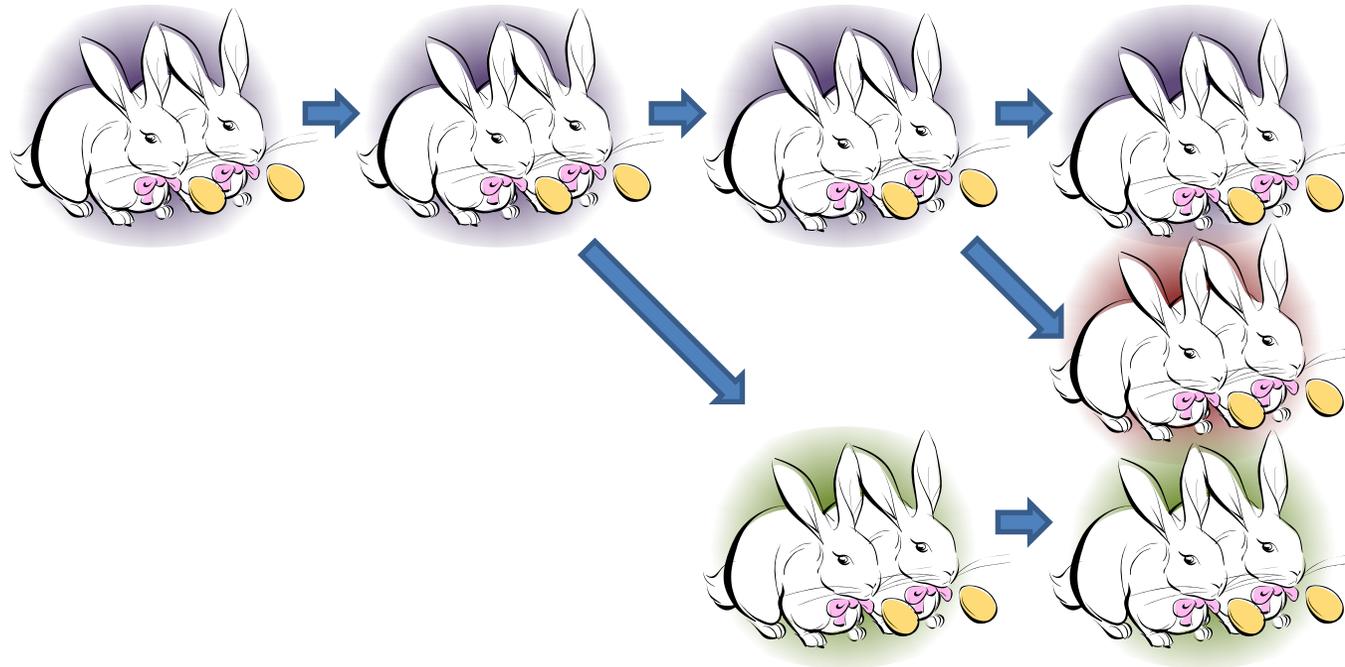
- What's the smallest number of moves needed to move the disks?
- Consider the following algorithm for moving k disks from the starting pole to the ending pole:
 1. (Recursively) transfer the top $k - 1$ disks from the starting pole to the temporary pole
 2. Move the bottom disk from the starting pole to the ending pole
 3. (Recursively) move the top $k - 1$ disks from the temporary pole to the ending pole

Tower of annoy

- How do we represent the running time of this algorithm recursively?
- We have to (recursively) move $k - 1$ disks, then a single disk, then (recursively) another $k - 1$ disks
- $m_k = m_{k-1} + 1 + m_{k-1}$ or
- $m_k = 2m_{k-1} + 1$
- Clearly, it takes 1 move to move a single disk, so $m_1 = 1$
- Find m_2 , m_3 , m_4 , and m_5

Fibonacci

- We all know and love Fibonacci by now
- Recall that it is supposed to model rabbit populations
- For the first month of their lives, they cannot reproduce
- After that, they reproduce every single month



Fibonacci rules

- From this information about rabbit physiology (which is a simplification, of course) we can think about pairs of rabbits
- At time k , rabbits born at time $k - 1$ will not reproduce
- Any rabbits born at $k - 2$ or earlier will, however
- So, we assume that all the rabbits from time $k - 2$ have doubled between time $k - 1$ and k
- Thus, our recurrence relation is:
 - $F_k = F_{k-1} + F_{k-2}, k \geq 2$
- Assuming one starting pair of rabbits, our initial conditions are:
 - $F_0 = 1$
 - $F_1 = 1$

Compound interest

- It's boring but useful
- Interest is compounded based on some period of time
- We can define the value recursively
- Let i is the **annual percentage rate** (APR) of interest
- Let m be the number of times per year the interest is compounded
- Thus, the total value of the investment at the k^{th} period is
 - $P_k = P_{k-1} + P_{k-1} \left(\frac{i}{m}\right), k \geq 1$
 - $P_0 =$ initial principle

Solving Recurrence Relations

Recursion

- ... is confusing
- We don't naturally think recursively (but perhaps you can raise your children to think that way?)
- As it turns out, the total number of moves needed to solve the Tower of Hanoi for n disks is $2^n - 1$
- Likewise, with an interest rate of i , a principle of P_0 , and m periods per year, the investment will yield $P_0 \left(\frac{i}{m} + 1 \right)^k$ after k periods

Finding explicit formulas by iteration

- Consequently, we want to be able to turn recurrence relations into explicit formulas whenever possible
- Often, the simplest way is to find these formulas by **iteration**
- The technique of iteration relies on writing out many expansions of the recursive sequence and looking for patterns
- That's it

Iteration example

- Find a pattern for the following recurrence relation:
 - $a_k = a_{k-1} + 2$
 - $a_0 = 1$
- Start at the first term
- Write the next below
- Do not combine like terms!
- Leave everything in expanded form until patterns emerge

Arithmetic sequence

- In principle, we should use mathematical induction to prove that the explicit formula we guess actually holds
- The previous example (odd integers) shows a simple example of an arithmetic sequence
- These are recurrences of the form:
 - $a_k = a_{k-1} + d$, for integers $k \geq 1$
- Note that these recurrences are always equivalent to
 - $a_n = a_0 + dn$, for all integers $n \geq 0$
- Let's prove it

Geometric sequence

- Find a pattern for the following recurrence relation:
 - $a_k = ra_{k-1}, k \geq 1$
 - $a_0 = a$
- Again, start at the first term
- Write the next below
- Do not combine like terms!
- Leave everything in expanded form until patterns emerge

Geometric sequence

- It appears that any geometric sequence with the following form
 - $a_k = ra_{k-1}, k \geq 1$
- is equivalent to
 - $a_n = a_0r^n$, for all integers $n \geq 0$
- This result applies directly to compound interest calculation
- Let's prove it

Employing outside formulas

- Sure, intelligent pattern matching gets you a long way
- However, it is sometimes necessary to substitute in some known formula to simplify a series of terms
- Recall
 - Geometric series: $1 + r + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$
 - Arithmetic series: $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

Tower of Hanoi solution

- Find a pattern for the following recurrence relation:
 - $m_k = 2m_{k-1} + 1, k \geq 2$
 - $m_1 = 1$
- Again, start at the first term
- Write the next below
- Do not combine like terms!
- Leave everything in expanded form until patterns emerge
- Use the arithmetic series or geometric series equations as needed

How many edges are in a complete graph?

- In a complete graph, every node is connected to every other node
- If we want to make a complete graph with k nodes, we can take a complete graph with $k - 1$ nodes, add a new node, and add $k - 1$ edges (so that all the old nodes are connected to the new node)
- Recursively, this means that the number of edges in a complete graph is
 - $s_k = s_{k-1} + (k - 1), k \geq 2$
 - $s_1 = 0$ (no edges in a graph with a single node)
- Use iteration to solve this recurrence relation

Mistakes were made!

- You might make a mistake when you are solving by iteration
- Consider the following recursive definition
 - $c_k = 2c_{k-1} + k, k \geq 1$
 - $c_0 = 1$
- Careless analysis might lead you to the explicit formula
 - $c_n = 2^n + n, n \geq 0$
- How would this be caught in a proof by induction verification?

Solving Second-Order Linear Homogeneous Relations with Constant Coefficients

Second-Order Linear Homogeneous Relations with Constant Coefficients

- Second-order linear homogeneous relations with constant coefficients are recurrence relations of the following form:
 - $a_k = Aa_{k-1} + Ba_{k-2}$ where $A, B \in \mathbb{R}$ and $B \neq 0$
- These relations are:
 - **Second order** because they depend on a_{k-1} and a_{k-2}
 - **Linear** because a_{k-1} and a_{k-2} are to the first power and not multiplied by each other
 - **Homogeneous** because there is no constant term
 - **Constant coefficients** because A and B are fixed values

Why do we care?

- I'm sure you're thinking that this is an awfully narrow class of recurrence relations to have special rules for
- It's true: There are many (infinitely many) ways to formulate a recurrence relation
 - Some have explicit formulas
 - Some do not have closed explicit formulas
- We care about this one partly for two reasons
 1. We can solve it
 2. It lets us get an explicit formula for Fibonacci!

Upcoming

Next time...

- Finish second order linear homogeneous recurrence relations with constant coefficients
- General recursion
- Review of set theory
 - Definitions
 - Properties of sets

Reminders

- Finish Assignment 2
 - **Due Monday**
- Read 6.1 and 6.2